Optimal Power Assignment to Minimize the Average Delay in Hybrid-ARQ Protocols

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***Abstract*— In this paper, the optimal power assignment strategy is determined for hybrid automatic-repeat-request (H-ARQ) protocols such that the average delay of the protocol is minimized for any given total transmission power budget and any targeted outage probability. A set of equations is derived that describe the optimal transmission power sequence and its optimality is shown based on the Karush-Kuhn-Tucker (KKT) Theorem. The set of equations enables an exact recursive calculation of the optimal transmission power per round, and the calculation complexity is fixed regardless of the maximum number of (re)transmission rounds allowed in the H-ARQ protocol. Compared to the conventional equal power assignment strategy, the optimal power assignment scheme achieves the same average delay with much less total transmission power. More importantly, for certain power budget levels, the optimal power assignment can make the H-ARQ protocol work while the equal power assignment cannot. Extensive numerical results are presented to illustrate the theoretical development.**

Index Terms:[1] **Hybrid automatic-repeat-request protocol, average delay, optimal power assignment, outage probability, Rayleigh fading.**

I. Introduction

Automatic-repeat-request (ARQ) protocol is an effective way to provide reliable transmission of data packets [1]–[5]. In basic ARQ protocols, a receiver decodes an information packet based only on the received signal in each (re)transmission round [1], [2]. In hybrid ARQ (H-ARQ) protocols, a receiver may decode an information packet by combining received signals from all previous (re)transmission rounds [3]–[5]. In general, H-ARQ protocols perform substantially better than basic ARQ protocols. The optimization of transmission power in retransmissions to save overall total power has been exploited for various ARQ protocols [6]–[9].

It is also important to optimize transmission power to reduce the average delay of ARQ protocols in practical systems (see for example [10]–[13] and the references therein). In [10], by assuming that partial CSI is available and with given delay constraint, optimal transmission power in each (re)transmission round was determined for an H-ARQ protocol by a linear programming method that selects a power value from a set of discrete power levels. In [11], to maximize the system throughput, an optimum number of retransmission rounds was determined in terms of block error probability. In [12], the joint statistics of transmission power and delay was investigated for a general ARQ protocol. A power-delay tradeoff curve was determined in terms of the data rate of each packet. Recently in [13], from an information-theoretic perspective, an optimal tradeoff between transmission power and packet queuing delay was characterized for communications over fading channels with feedbacks.

Note that, while the delay analysis has been well understood for ARQ protocols [10]–[13], the power assignment in retransmissions to minimize the average delay of H-ARQ protocols turns out to be very challenging, which is tackled in this paper. Specifically, we consider H-ARQ transmission protocols in which a destination node decodes an information packet by combining all received signals from previous (re)transmission rounds to increase detection reliability. We assume that the source-destination channel experiences *quasi-static* Rayleigh fading. Our goal is to determine an optimal power assignment strategy that minimizes the average delay of the H-ARQ protocol with any given total transmission power budget and any targeted outage probability. First, we derive a set of equations that describe the optimal transmission power sequence and prove its optimality based on the Karush-Kuhn-Tucker (KKT) Theorem [15]. The set of equations enables an exact recursive calculation of the optimal transmission power per round, and the complexity is fixed regardless of the number of (re)transmissions allowed in the H-ARQ protocol. Compared to the conventional equal power assignment scheme, the optimal power assignment strategy achieves the same average delay with much less total transmission power. As shown in the numerical results, to achieve an average delay of $\bar{D} = 1.05$, the optimal power assignment strategy needs a transmission power budget of $P_{tot} = 250$ while the equal power assignment scheme needs a total transmission power budget of $P_{tot} = 400$. More importantly, the optimal power assignment scheme can achieve certain delay performance for some given transmission power budget, which is impossible for the equal power assignment scheme. Substantial numerical results are presented to illustrate the theoretical development.

II. System Model and Problem Formulation

We consider an H-ARQ transmission protocol implemented between one source and one destination allowing maximum $L-1$ retransmissions for each information package, as illustrated in Fig. 1. The operation of the H-ARQ transmission scheme is described as follows. First, the source transmits an information packet to the destination and the destination indicates success or failure of receiving the packet by feeding back a single bit of acknowledge (ACK) or negative-

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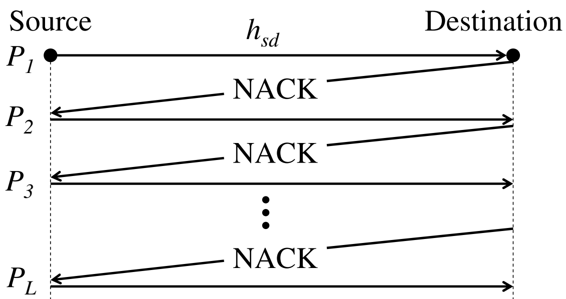

Fig. 1. Illustration of a hybrid-ARQ protocol with transmission power P_l per (re-)transmission round, $1 \leq l \leq L$.

acknowledgement (NACK), respectively. The feedback channel is assumed to be error-free. Then, if a NACK is received by the source and the maximum number of retransmissions is not reached, the source retransmits the packet at a potentially different transmission power to be determined/optimized. If an ACK is received by the source or the maximum number of retransmissions is reached, the source begins transmission of a new information packet. In each (re)transmission round, the destination attempts to decode an information packet by combining received signals from all previous (re)transmission rounds based on the maximal-ratio-combining (MRC) technique [14]. If the destination still cannot decode an information packet after L (re)transmission rounds, then an outage is declared which means that the signal-to-noise ratio (SNR) of the combined received signals at the destination is below a required SNR.

The H-ARQ transmission scheme can be modeled as follows. The received signal $y_{sd,l}$ at the destination at the l-th (re)transmission round can be written as

$$y_{sd,l} = \sqrt{P_l}\ h_{sd}\ x_s + \eta_{sd,l}, \quad l = 1, 2, \cdots, L, \qquad (1)$$

where x_s is the transmitted information symbol from the source, P_l is the transmission power used by the source at the l-th transmission round, h_{sd} is the source-destination channel coefficient, and $\eta_{sd,l}$ is additive noise. The channel coefficient h_{sd} is modeled as a zero-mean complex Gaussian random variable with variance σ_{sd}^2. The channel is assumed to be *quasi-static*, i.e., the channel remains fixed during (re)transmissions of the same packet and may change independently when transmitting a new information packet. The source-destination channel is assumed to be known at the destination side, but unknown at the source side. The additive noise $\eta_{sd,l}$ in each transmission round is modeled as a zero-mean complex Gaussian random variable with variance $\mathcal{N}_0$.

At the destination, it combines the received signals from all previous (re)transmission rounds and jointly decodes the information packet. The overall SNR of the combined signal at the destination at the l-th $(1 \leq l \leq L)$ (re)transmission round is

$$\gamma_{sd,l} = \frac{\sum_{i=1}^{l} P_i |h_{sd}|^2}{\mathcal{N}_0}. \qquad (2)$$

Thus, for a targeted SNR γ_0, the probability of the event that the destination node cannot decode correctly after l (re)transmissions can be calculated, for a Rayleigh fading channel with variance σ_{sd}^2, to be

$$p^{out,l} = \Pr\left[\gamma_{sd,l} < \gamma_0\right] = 1 - e^{-\frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 \sum_{i=1}^{l} P_i}}. \qquad (3)$$

Set $p^{out,0} = 1$. Then, the probability that the H-ARQ protocol stops at the l-th, $1 \leq l < L$, (re)transmission round is $p^{out,l-1} - p^{out,l}$, which means the destination cannot decode correctly at the $(l-1)$-th round, but succeeds at the l-th round.

The goal of this paper is to find an optimal transmission power sequence $\mathbf{P} = [P_1, P_2, ..., P_L]$ for the H-ARQ protocol such that the average delay to deliver an information packet is minimized under a given power budget and a targeted outage probability performance. Since the probability that the protocol succeeds exactly at the l-th round is $p^{out,l-1} - p^{out,l}$ and the corresponding total transmission power is $P_1 + P_2 + \cdots + P_l$, the average total transmission power and average delay of the H-ARQ protocol, denoted as $\bar{P}$ and $\bar{D}$ respectively, can be given by

$$\bar{P} = \sum_{l=1}^{L-1} \left(p^{out,l-1} - p^{out,l}\right) \sum_{i=1}^{l} P_i + p^{out,L-1} \sum_{i=1}^{L} P_i, \qquad (4)$$

$$\bar{D} = \sum_{l=1}^{L-1} \left(p^{out,l-1} - p^{out,l}\right) l + p^{out,L-1} L. \qquad (5)$$

For the H-ARQ protocol with a targeted outage probability p_0 and a given total transmission power budget P_{tot}, the problem of finding optimal power assignment to minimize the average delay can be formulated as

$$\begin{aligned} &\min \quad \bar{D} \quad \text{with respect to } P_1, P_2, \cdots, P_L \geq 0 \\ &\text{subject to} \quad \begin{cases} \bar{P} \leq P_{tot} \\ p^{out,L} \leq p_0 \end{cases} \end{aligned} \qquad (6)$$

where $\bar{P}$ and $\bar{D}$ are specified in (4) and (5), respectively.

III. Optimal Transmission Power Assignment

A. Simplify the Optimization Problem

We first simplify the optimization problem in (6), then we will solve it based on the KKT Theorem. The average total transmission power in (4) can be rewritten by switching the summation order as follows

$$\bar{P} = \sum_{i=1}^{L} P_i \left[\sum_{l=i}^{L-1} \left(p^{out,l-1} - p^{out,l}\right) + p^{out,L-1}\right], \qquad (7)$$

where we consider the summation in terms of the index i, $1 \leq i \leq L$, first. Since $\sum_{l=i}^{L-1} \left(p^{out,l-1} - p^{out,l}\right) = p^{out,i-1} - p^{out,L-1}$, the average total transmission power can be simplified to

$$\bar{P} = P_1 + \sum_{l=2}^{L} P_l\ p^{out,l-1}. \qquad (8)$$

Similarly, the average delay in (5) can be simplified to

$$\bar{D} = 1 + \sum_{l=2}^{L} p^{out,l-1}. \qquad (9)$$

Note that the first constraint in (6) means that the average total transmission power cannot exceed the given power budget P_{tot}, i.e.,

$$P_1 + \sum_{l=2}^{L} P_l \; p^{out,l-1} \leq P_{tot}. \quad (10)$$

The second constraint in (6) implies that with a targeted SNR γ_0, the outage probability of the H-ARQ protocol with L (re)transmissions should be no more than the specified outage probability value p_0, i.e.,

$$p^{out,L} = 1 - e^{-\frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 \sum_{i=1}^{L} P_i}} \leq p_0. \quad (11)$$

Define $P_0 \triangleq \frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 \ln \frac{1}{1-p_0}}$, then the constraint takes the form

$$\sum_{l=1}^{L} P_l \geq P_0. \quad (12)$$

From (9), (10) and (12), the minimization problem in (6) can be reformulated as follows:

$$\min_{P_1,\cdots,P_L \geq 0} \quad \bar{D}(\mathbf{P}) = 1 + \sum_{l=2}^{L} \left(1 - e^{-\frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 \sum_{i=1}^{l-1} P_i}}\right) \quad (13)$$

$$\text{subject to} \begin{cases} P_1 + \sum_{l=2}^{L} P_l \left(1 - e^{-\frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 \sum_{i=1}^{l-1} P_i}}\right) \leq P_{tot} \\ \sum_{l=1}^{L} P_l \geq P_0 \end{cases}$$

where $\mathbf{P} = [P_1, P_2, \cdots, P_L]$ is the power sequence.

Next, we prove that an optimal power sequence $\mathbf{P} = [P_1, P_2, \cdots, P_L]$ that minimizes the average delay $\bar{D}(\mathbf{P})$ in (13) can be found at the boundary of the first constraint (i.e., the first constraint in (13) holds with equality). If there exists a power sequence $P_1^*, P_2^*, \cdots, P_L^*$ such that the average delay $\bar{D}$ is minimized and

$$\begin{cases} P_1^* + \sum_{l=2}^{L} P_l^* \left(1 - e^{-\frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 \sum_{i=1}^{l-1} P_i^*}}\right) < P_{tot} \\ \sum_{l=1}^{L} P_l^* \geq P_0 \end{cases}$$

then let us consider another power sequence

$$\begin{cases} \tilde{P}_l = P_l^*, & 1 \leq l \leq L-1; \\ \tilde{P}_L = \eta P_L^*, & l = L, \end{cases}$$

where η is given by

$$\eta \triangleq \frac{1}{P_L^*} \left(1 - e^{-\frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 \sum_{i=1}^{L-1} P_i^*}}\right)^{-1} \times \left[P_{tot} - P_1^* - \sum_{l=2}^{L-1} P_l^* \left(1 - e^{-\frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 \sum_{i=1}^{l-1} P_i^*}}\right)\right]. \quad (14)$$

We can see that the new power sequence $\tilde{P}_1, \tilde{P}_2, \cdots, \tilde{P}_L$ satisfies the first constraint in (13) with equality, i.e.,

$$\bar{P}(\tilde{P}_1, \tilde{P}_2, \cdots, \tilde{P}_L) = P_1^* + \sum_{l=2}^{L-1} P_l^* \left(1 - e^{-\frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 \sum_{i=1}^{l-1} P_i^*}}\right) + \eta P_L^* \left(1 - e^{-\frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 \sum_{i=1}^{L-1} P_i^*}}\right) = P_{tot}. \quad (15)$$

Since $P_{tot} - P_1^* - \sum_{l=2}^{L-1} P_l^* \left(1 - e^{-\frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 \sum_{i=1}^{l-1} P_i^*}}\right) > P_L^* \left(1 - e^{-\frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 \sum_{i=1}^{L-1} P_i^*}}\right)$, η is lower bounded as

$$\eta > \frac{1}{P_L^*} \left(1 - e^{-\frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 \sum_{i=1}^{L-1} P_i^*}}\right)^{-1} P_L^* \left(1 - e^{-\frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 \sum_{i=1}^{L-1} P_i^*}}\right) = 1.$$

Thus, we have

$$\sum_{l=1}^{L} \tilde{P}_l = \sum_{l=1}^{L-1} P_l^* + \eta P_L^* > \sum_{l=1}^{L} P_l^* \geq P_0, \quad (17)$$

which means the second constraint of (13) is also satisfied with the new power sequence. Since $\tilde{P}_l = P_l^*$ for $1 \leq l \leq L-1$, we can see that

$$\bar{D}(\tilde{P}_1, \tilde{P}_2, \cdots, \tilde{P}_L) = \bar{D}(P_1^*, P_2^*, \cdots, P_L^*), \quad (18)$$

i.e., the average delay resulting from the new power sequence $\tilde{P}_1, \tilde{P}_2, \cdots, \tilde{P}_L$ is the same as that with the power sequence $P_1^*, P_2^*, \cdots, P_L^*$. This means the new power sequence is also an optimal solution that minimizes the average delay $\bar{D}$ in (13) and it is located at the boundary of the first constraint in (13) (with equality), i.e.,

$$P_1 + \sum_{l=2}^{L} P_l \left(1 - e^{-\frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 \sum_{i=1}^{l-1} P_i}}\right) = P_{tot}. \quad (19)$$

As a consequence, the transmission power in the last round P_L can be determined as

$$P_L = \left[P_{tot} - P_1 - \sum_{l=2}^{L-1} P_l \left(1 - e^{-\frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 \sum_{i=1}^{l-1} P_i}}\right)\right] \times \left(1 - e^{-\frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 \sum_{i=1}^{L-1} P_i}}\right)^{-1}. \quad (20)$$

From the second constraint in (13), the above P_L must satisfy

$$P_L \geq P_0 - \sum_{l=1}^{L-1} P_l. \quad (21)$$

Combining (20) and (21), we have a new constraint

$$\left(P_0 - \sum_{l=1}^{L-1} P_l\right)\left(1 - e^{-\frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 \sum_{i=1}^{L-1} P_i}}\right) \leq P_{tot} - P_1 - \sum_{l=2}^{L-1} P_l \left(1 - e^{-\frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 \sum_{i=1}^{l-1} P_i}}\right). \quad (22)$$

Therefore, the minimization problem in (13) is simplified as:

$$\min_{P_1,\cdots,P_L \geq 0} \quad \bar{D}(\mathbf{P}) = 1 + \sum_{l=2}^{L} \left(1 - e^{-\frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 \sum_{i=1}^{l-1} P_i}}\right) \quad (23)$$

$$\text{subject to} \quad P_1 + \sum_{l=2}^{L-1} P_l \left(1 - e^{-\frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 \sum_{i=1}^{l-1} P_i}}\right) + \left(P_0 - \sum_{l=1}^{L-1} P_l\right)\left(1 - e^{-\frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 \sum_{i=1}^{L-1} P_i}}\right) - P_{tot} \leq 0,$$

with only one constraint instead of two in (13).

B. Solve the Problem based on the KKT Theorem

In the following, we try to solve the minimization problem in (23) based on the KKT Theorem. Let us denote

$$g_0(\mathbf{P}) \triangleq P_1 + \sum_{l=2}^{L-1} P_l\left(1 - e^{-\frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 \sum_{i=1}^{l-1} P_i}}\right) + \left(P_0 - \sum_{l=1}^{L-1} P_l\right)\left(1 - e^{-\frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 \sum_{i=1}^{L-1} P_i}}\right) - P_{tot}, \quad (24)$$

and $g_i(\mathbf{P}) \triangleq -P_i$ for any $i = 1, 2, \cdots, L$. Then, the minimization problem in (23) can be expressed as

$$\min \quad \bar{D}(\mathbf{P}) = 1 + \sum_{l=2}^{L}\left(1 - e^{-\frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 \sum_{i=1}^{l-1} P_i}}\right) \quad (25)$$
$$\text{subject to} \quad g_i(\mathbf{P}) \leq 0, \qquad i = 0,\, 1,\, 2,\, \cdots,\, L.$$

The KKT multiplier function of the above minimization is [15]

$$\mathcal{L}\left(\mathbf{P}; \lambda_0, \lambda_1, \cdots, \lambda_L\right) = \bar{D}(\mathbf{P}) + \sum_{i=0}^{L} \lambda_i\, g_i(\mathbf{P}), \quad (26)$$

where $\lambda_0,\, \lambda_1,\, \cdots,\, \lambda_L$ are real numbers. According to the KKT Theorem ([15], p.458), if a power sequence $\mathbf{P}^* = [P_1^*,\, P_2^*,\, \cdots,\, P_L^*]$ is an optimal solution to the minimization problem (25), there must exist constants $\lambda_0^*,\, \lambda_1^*,\, \cdots,\, \lambda_L^*$ such that

$$\left.\frac{\partial \mathcal{L}\left(\mathbf{P};\, \lambda_0^*,\, \lambda_1^*,\, \cdots,\, \lambda_L^*\right)}{\partial P_k}\right|_{\mathbf{P}=\mathbf{P}^*} = 0, \quad k = 1, 2, \cdots, L; \quad (27)$$
$$\lambda_i^*\, g_i(\mathbf{P}^*) = 0, \quad i = 0, 1, \cdots, L; \quad (28)$$
$$\lambda_i^* \geq 0, \quad i = 0, 1, \cdots, L. \quad (29)$$

The condition in (28) is called a complementary slackness condition [15]. We say a constraint $g_i(\mathbf{P}) \leq 0$ is inactive if $g_i(\mathbf{P}^*) < 0$, in this case the corresponding parameter λ_i^* must be zero according to the condition in (28). On the other hand, if $\lambda_i^* > 0$, then $g_i(\mathbf{P}^*)$ must be zero, i.e., the constraint $g_i(\mathbf{P}) \leq 0$ is active in the minimization problem.

First, we want to show that $g_0(\mathbf{P}^*)$ must be zero. If $g_0(\mathbf{P}^*) < 0$, then according to the complementary slackness condition in (28), λ_0^* must be zero. In this case, for any $k = 1,\, 2,\, \cdots,\, L$,

$$\left.\frac{\partial \mathcal{L}\left(\mathbf{P};\, \lambda_0^*,\, \lambda_1^*,\, \cdots,\, \lambda_L^*\right)}{\partial P_k}\right|_{\mathbf{P}=\mathbf{P}^*} = -\sum_{l=k+1}^{L} \frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2\left(\sum_{i=1}^{l-1} P_i^*\right)^2}\, e^{-\frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 \sum_{i=1}^{l-1} P_i^*}} - \lambda_k^* < 0,$$

which is contradictory to the condition (27). Thus, $g_0(\mathbf{P}^*) = 0$, which means that the constraint $g_0(\mathbf{P}) \leq 0$ is active with equality in the minimization. Moveover, with $g_0(\mathbf{P}^*) = 0$, we know that

$$P_L^* = P_0 - \sum_{l=1}^{L-1} P_l^*. \quad (30)$$

Next, we consider the case that none of the power P_k^* ($k = 1,\, 2,\, \cdots,\, L$) is zero, i.e., each of them is strictly positive. In this case, according to the condition in (28), we know that $\lambda_k^* = 0$ for any $k = 1,\, 2,\, \cdots,\, L$. Thus, for any $k = 1,\, 2,\, \cdots,\, L-1$, we have

$$\left.\frac{\partial \mathcal{L}\left(\mathbf{P};\, \lambda_0^*\right)}{\partial P_k}\right|_{\mathbf{P}=\mathbf{P}^*} = -\sum_{l=k+1}^{L} \frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2\left(\sum_{i=1}^{l-1} P_i^*\right)^2}\, e^{-\frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 \sum_{i=1}^{l-1} P_i^*}} + \lambda_0^*\left[\left(1 - e^{-\frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 \sum_{i=1}^{k-1} P_i^*}}\right) - \left(1 - e^{-\frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 \sum_{i=1}^{L-1} P_i^*}}\right) - \sum_{l=k+1}^{L-1} \frac{P_l \gamma_0 \mathcal{N}_0}{\sigma_{sd}^2\left(\sum_{i=1}^{l-1} P_i^*\right)^2}\, e^{-\frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 \sum_{i=1}^{l-1} P_i^*}} - \frac{\left(P_0 - \sum_{l=i}^{L-1} P_i^*\right)\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2\left(\sum_{i=1}^{L-1} P_i^*\right)^2}\, e^{-\frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 \sum_{i=1}^{L-1} P_i^*}}\right].$$

With $\left.\frac{\partial \mathcal{L}}{\partial P_1}\right|_{\mathbf{P}=\mathbf{P}^*} = 0$ and $\left.\frac{\partial \mathcal{L}}{\partial P_2}\right|_{\mathbf{P}=\mathbf{P}^*} = 0$, we have

$$\left.\frac{\partial \mathcal{L}}{\partial P_1}\right|_{\mathbf{P}=\mathbf{P}^*} - \left.\frac{\partial \mathcal{L}}{\partial P_2}\right|_{\mathbf{P}=\mathbf{P}^*} = -\frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 {P_1^*}^2}\, e^{-\frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 P_1^*}} + \lambda_0^*\left[1 - \frac{P_2^* \gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 {P_1^*}^2}\right] e^{-\frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 P_1^*}} = 0,$$

which means

$$P_2^* = \frac{\sigma_{sd}^2 {P_1^*}^2}{\gamma_0 \mathcal{N}_0} - \frac{1}{\lambda_0^*}. \quad (31)$$

For any $k = 3, 4, \cdots, L-1$, from $\left.\frac{\partial \mathcal{L}}{\partial P_{k-1}}\right|_{\mathbf{P}=\mathbf{P}^*} = 0$ and $\left.\frac{\partial \mathcal{L}}{\partial P_k}\right|_{\mathbf{P}=\mathbf{P}^*} = 0$, we have

$$\left.\frac{\partial \mathcal{L}}{\partial P_{k-1}}\right|_{\mathbf{P}=\mathbf{P}^*} - \left.\frac{\partial \mathcal{L}}{\partial P_k}\right|_{\mathbf{P}=\mathbf{P}^*} = -\frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2\left(\sum_{i=1}^{k-1} P_i^*\right)^2}\, e^{-\frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 \sum_{i=1}^{k-1} P_i^*}} + \lambda_0^*\left[e^{-\frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 \sum_{i=1}^{k-1} P_i^*}} - e^{-\frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 \sum_{i=1}^{k-2} P_i^*}} - \frac{P_k \gamma_0 \mathcal{N}_0}{\sigma_{sd}^2\left(\sum_{i=1}^{k-1} P_i^*\right)^2}\, e^{-\frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 \sum_{i=1}^{k-1} P_i^*}}\right] = 0,$$

which implies

$$P_k^* = \frac{\sigma_{sd}^2 (\sum_{i=1}^{k-1} P_i^*)^2}{\gamma_0 \mathcal{N}_0}\left[1 - e^{-\frac{P_{k-1}^* \gamma_0 \mathcal{N}_0}{\sigma_{sd}^2\left(\sum_{i=1}^{k-1} P_i^*\right)\left(\sum_{i=1}^{k-2} P_i^*\right)}}\right] - \frac{1}{\lambda_0^*}. \quad (32)$$

Finally, according to $\left.\frac{\partial \mathcal{L}}{\partial P_{L-1}}\right|_{\mathbf{P}=\mathbf{P}^*} = 0$, we have

$$P_0 - \sum_{k=1}^{L-1} P_k^* = \frac{\sigma_{sd}^2 (\sum_{i=1}^{L-1} P_i^*)^2}{\gamma_0 \mathcal{N}_0} \times \left[1 - e^{-\frac{P_{L-1}^* \gamma_0 \mathcal{N}_0}{\sigma_{sd}^2\left(\sum_{i=1}^{L-1} P_i^*\right)\left(\sum_{i=1}^{L-2} P_i^*\right)}}\right] - \frac{1}{\lambda_0^*}. \quad (33)$$

Combining (30) and (33), we can determine P_L^* by

$$P_L^* = \frac{\sigma_{sd}^2 (\sum_{i=1}^{L-1} P_i^*)^2}{\gamma_0 \mathcal{N}_0}\left[1 - e^{-\frac{P_{L-1}^* \gamma_0 \mathcal{N}_0}{\sigma_{sd}^2\left(\sum_{i=1}^{L-1} P_i^*\right)\left(\sum_{i=1}^{L-2} P_i^*\right)}}\right] - \frac{1}{\lambda_0^*}. \quad (34)$$

We summarize the above discussion in the following theorem which provides a recursive way to calculate the optimal power sequence $P_1^*, P_2^*, \cdots, P_L^*$.

Theorem 1: In the H-ARQ transmission protocol, with given power budget P_{tot} and targeted outage probability p_0, the optimal transmission power $P_1^*, P_2^*, \cdots, P_L^*$ that minimizes the average delay can be determined as follows:

$$P_2^* = \frac{\sigma_{sd}^2 {P_1^*}^2}{\gamma_0 \mathcal{N}_0} - \frac{1}{\lambda_0^*}, \tag{35}$$

$$P_k^* = \frac{\sigma_{sd}^2 (\sum_{i=1}^{k-1} P_i^*)^2}{\gamma_0 \mathcal{N}_0} \left[1 - e^{-\frac{P_{k-1}^* \gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 \left(\sum_{i=1}^{k-1} P_i^*\right)\left(\sum_{i=1}^{k-2} P_i^*\right)}} \right] - \frac{1}{\lambda_0^*}, \tag{36}$$

for $k = 3, 4, \cdots, L$, and

$$P_1^* + \sum_{k=2}^{L} P_k^* \left[1 - e^{-\frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 \sum_{i=1}^{k-1} P_i^*}} \right] = P_{tot}, \tag{37}$$

$$P_1^* + P_2^* + \cdots + P_L^* = P_0, \tag{38}$$

where $P_0 \triangleq \frac{\gamma_0 \mathcal{N}_0}{\sigma_{sd}^2 \ln \frac{1}{1-p_0}}$. □

From Theorem 1, we observe that for any fixed $\lambda_0^* > 0$ and P_1^*, all other transmission power $P_k^*, k = 2, 3, \cdots, L$, can be determined by the equations in (35) and (36) accordingly. Thus, we can search parameters λ_0^* and P_1^* based on the constraints in (37) and (38) to find the optimal solution that minimizes the average delay. We can see that in this way, the complexity is reduced to the search of two variables λ_0^* and P_1^*, and the complexity is fixed no matter how large the maximum number of (re)transmission rounds L is.

Note that we summarize the result in Theorem 1 for the case that each power P_k^* $(1 \leq k \leq L)$ is strictly positive, i.e., nonzero for all of the L transmission rounds. If the equations in the theorem do not result in strictly positive power sequence, then we need to consider the case that the optimal transmission power sequence may have zero values in some transmission rounds. In this case, we can show that the optimal power value in the last transmission round must be zero, i.e., $P_L^* = 0$. The proof is detailed as follows. If there exists a power sequence $P_1^*, P_2^*, \cdots, P_L^*$ having a zero element $P_{k_0}^* = 0$ for some k_0 $(1 \leq k_0 \leq L-1)$ and $P_L^* \neq 0$ which is an optimal solution of the minimization problem in (25). Then, we consider another power sequence

$$\tilde{P}_k = \begin{cases} P_k^*, & 1 \leq k \leq k_0 - 1; \\ P_{k+1}^*, & k_0 \leq k \leq L-1; \\ 0 & k = L. \end{cases} \tag{39}$$

We can see that

$$\sum_{k=1}^{L} \tilde{P}_k = \sum_{k=1}^{L} P_k^* = P_0, \tag{40}$$

$$\bar{P}\left(\tilde{P}_1, \tilde{P}_2, \cdots, \tilde{P}_L\right) = \bar{P}\left(P_1^*, P_2^*, \cdots, P_L^*\right) = P_{tot}, \tag{41}$$

i.e., the new power sequence also satisfies the two constraints in the minimization problem. The outage probability performances of the two power sequences are the same. The only difference between the two sequences is that the new sequence stops transmission at the last round while the previous one holds transmission in the middle (in the k_0-th round), so

$$\bar{D}\left(P_1^*, P_2^*, \cdots, P_L^*\right) = \bar{D}\left(\tilde{P}_1, \tilde{P}_2, \cdots, \tilde{P}_L\right) + 1, \tag{42}$$

which means that the new power sequence would result in a smaller average delay. This is contradictory to the assumption that the power sequence $P_1^*, P_2^*, \cdots, P_L^*$ with $P_L^* \neq 0$ is an optimal solution to minimize the average delay. By repeating the above argument, we can conclude that if some of the optimal transmission power values are zeros in some transmission rounds (say totally L_0 rounds), then it must happen in the last L_0 transmission rounds, i.e., $P_{L-L_0+1}^* = \cdots = P_L^* = 0$. In this case, we can reduce the minimization problem by considering an H-ARQ protocol with $L - L_0$ transmission rounds, and then utilize Theorem 1 again by replacing L with $L - L_0$ to obtain the strictly positive power sequence $P_1^*, P_2^*, \cdots, P_{L-L_0}^*$.

TABLE I
OPTIMAL POWER SEQUENCE TO MINIMIZE THE AVERAGE DELAY WITH DIFFERENT TOTAL POWER BUDGETS ($L = 5$ AND $p_0 = 10^{-1}$).

P_1^*	P_2^*	P_3^*	P_4^*	P_5^*	P_{tot}
15.20	15.57	19.26	21.89	22.99	35
23.50	28.17	28.20	15.10	0	40
30.80	39.04	25.15	0	0	45
37.90	48.67	8.35	0	0	50
45.10	49.72	0	0	0	55
77.90	16.92	0	0	0	80
94.90	0.01	0	0	0	95

IV. NUMERICAL RESULTS

In this section, we present some numerical results to illustrate the optimal power sequence obtained from Theorem 1 and also compare the average delay of the H-ARQ protocols with and without the optimal power assignment strategy. In the numerical calculation, we assume that the variance of the channel h_{sd} is $\sigma_{sd}^2 = 1$, the noise variance is $\mathcal{N}_0 = 1$, and the required SNR for reliable decoding is $\gamma_0 = 10$ dB.

In Table I, we list optimal power sequences of the H-ARQ protocol with different total power budgets P_{tot}. We assume that the H-ARQ protocol has maximum $L = 5$ (re)transmission rounds and the targeted outage probability is $p_0 = 10^{-1}$. We can see that when the power budget is $P_{tot} = 35$, the optimal power sequence has strictly positive value in each transmission round. When the power budget is $P_{tot} = 40$ and larger, the optimal power sequence has some zero power at the last few transmission rounds. It is also interesting to observe that the optimal power sequence is increasing in the first few transmission rounds (up to certain level which depends on the power budget) and then decreasing after that.

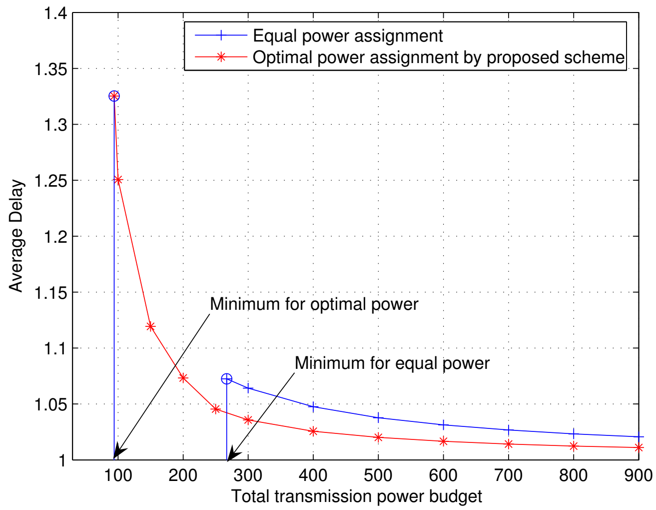

Fig. 2. Comparison of the average delay of the H-ARQ protocol with the equal and optimal power assignment strategies, $\gamma_0 = 10$ dB, $p_0 = 10^{-2}$, $L = 4$.

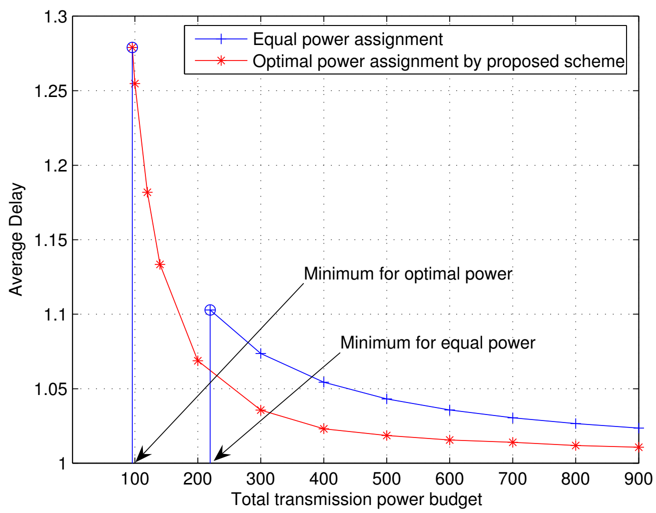

Fig. 3. Comparison of the average delay of the H-ARQ protocol with the equal and optimal power assignment strategies, $\gamma_0 = 10$ dB, $p_0 = 10^{-2}$, $L = 5$.

We compare the average delay of the H-ARQ protocol with the equal and optimal power assignment strategies in Figs. 2 and 3 for $L = 4$ and $L = 5$, respectively. From both figures, we can see that compared to the equal power scheme, the optimal power assignment scheme achieves the same average delay with much less power budget. For example, in Fig. 2 ($L = 4$), to achieve the average delay $\bar{D} = 1.05$, the optimal power assignment needs power budget of $P_{tot} = 250$ while the equal power assignment needs power budget of $P_{tot} = 400$. Moreover, the minimum power budget required in the equal power scheme is $P_{tot} = 270$ while it is only $P_{tot} = 90$ for the optimal power scheme. For a given power budget P_{tot} in the range of $90 \sim 250$, the optimal power assignment can make the H-ARQ protocol work while the equal power assignment cannot. We have similar observations in Fig. 3 for $L = 5$.

V. Conclusion

In this paper, we determined the optimal power assignment strategy to minimize the average delay of the H-ARQ communication protocol over quasi-static Rayleigh fading channels with any given power budget and any targeted outage probability. We derived a set of equations that describe the optimal transmission power sequence and proved its optimality based on the KKT Theorem. The set of equations enables an exact recursive calculation of the optimal transmission power per round, and the calculation complexity is fixed regardless of the maximum number of (re)transmission rounds allowed in the H-ARQ protocol. Compared to the conventional equal power assignment scheme, the optimal power assignment scheme achieves the same average delay with much less total transmission power. Moreover, the minimum power budget required in the optimal power assignment scheme is much less than that in the equal power assignment scheme. For certain power budget levels, the optimal power assignment strategy can make the H-ARQ protocol work while the equal power assignment strategy cannot.

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